Formulae

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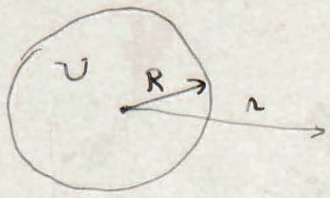
meeting Aug 22nd 1941

I) For Spherical Symetry:

A - Diffusion equation:

$$D(c)\,\frac{d^2(r\rho)}{dr^2} - S(c)\,r\rho + Q(r)\,r = 0 \quad \ldots (1)$$

$$D(c) = \frac{v\,\lambda(c)}{3}$$



ρ = density of thermal neutrons in C

$$S(c) = \frac{v}{\lambda(c)}\,\frac{\sigma_{abs.}(c)}{\sigma_{scat}(c)}$$

$$\frac{dQ}{dr} = 0 \qquad \rho(R) = 0$$

Solution:

$$\rho(r) = \frac{Q}{S}\left(1 - \frac{R}{r}\,e^{-\frac{(r-R)}{A}}\right) \ldots (2); \qquad A = \sqrt{\frac{D}{S}} = \frac{\lambda(c)}{3}\sqrt{\frac{\sigma_{scat}}{\sigma_{abs}}} = \frac{\lambda(c)}{3}\sqrt{k^{th}} \cong 45.4\text{ cm.}$$

$$I^{th} = D(c)\,4\pi R^2\,\frac{d\rho(R)}{dr} \quad \ldots (3)$$

for black spheres:

$$B = \frac{\lambda^{res}}{\sqrt{3}}\sqrt{k^{res}} \cong 6.5\text{ cm.}$$

$$I_0^{th} = 4\pi Q R A^2 (1 + R/A)$$

$$I^{res} = 4\pi Q R B^2 (1 + R/B)$$

$$\varepsilon = \frac{\text{no. of thermal neutrons absorbed}}{\text{no. of resonance neutrons absorbed}}$$

for black spheres:

$$\varepsilon_0 = \frac{A^2}{B^2}\,\frac{1 + R/A}{1 + R/B} \quad \ldots (4) \qquad \varepsilon = \varepsilon_0\,\varphi \qquad I^{th} = I_0\,\varphi$$

Inside the Uranium Spheres:

$$D(U) = \frac{v\,\lambda(U)}{3}$$

$$S(U) = v\,N_u\,\sigma_a(U)$$

$$D(U)\,\frac{\partial^2(\rho r)}{\partial r^2} - S(U)\,r\rho = 0 \quad \ldots (5)$$

connecting the solutions at the boundary:

$$\varphi = \frac{\frac{\lambda_{sc}(U)}{\mathcal{U}}\left\{\frac{e^{R/\mathcal{U}} + e^{-R/\mathcal{U}}}{e^{R/\mathcal{U}} - e^{-R/\mathcal{U}}} - \frac{\mathcal{U}}{R}\right\}}{\frac{\lambda_{sc}(c)}{R}(1 + R/A) + \frac{\lambda_{sc}(U)}{\mathcal{U}}\left\{\frac{e^{R/\mathcal{U}} + e^{-R/\mathcal{U}}}{e^{R/\mathcal{U}} - e^{-R/\mathcal{U}}} - \frac{\mathcal{U}}{R}\right\}} \quad \ldots (6)$$

$$\mathcal{U} = \lambda(U)\sqrt{\frac{\sigma_a(U)}{3\sigma_{abs}(U)}}$$

$$\varepsilon = \left\{\frac{A^2}{B^2}\right\}\left\{\frac{1}{1 + R/B} \times \frac{1}{\left(\frac{\lambda(c)}{RG\sqrt{\frac{3\sigma_a(U)}{\sigma_{sc}(U)}} - \lambda(U)} + \frac{1}{1 + R/A}\right)}\right\} \quad \ldots (7)$$

$$G = \frac{e^{R/\mathcal{U}} + e^{-R/\mathcal{U}}}{e^{R/\mathcal{U}} - e^{-R/\mathcal{U}}}$$

for $R/\mathcal{U} > 2$
$G \cong 1$

q = no of usefully absorbed neutrons

$$q = \frac{\varepsilon\alpha}{1 + \varepsilon\alpha}(1 - \alpha) \quad \ldots (8)$$

$$\alpha = \frac{\text{average thermal neutron density in lattice}}{\text{thermal neutron density in the absence of U}}$$

1.) $k^{res} = 6.5 \ln \frac{E_2}{E_1}$ for $\frac{E_2}{E_1} = 10$ $k^{res} = 15$ $\frac{\lambda^{res}(C)}{\lambda(C)} = 1.18$

$B = \frac{\lambda^{res}}{\sqrt{3}} \sqrt{k^{res}} = 6.5\,\mathrm{cm}$ for graphite of density 1.7

2.) $k^{th} = \frac{\sigma_{sc}}{\sigma_c}$ $\sigma_{sc} = 4.8$ $\sigma_c < 10^{-2}$ $k^{th} > 480$

3.) Single sphere: $D \frac{d^2(r\rho)}{dr^2} - S(r\rho) + Q(r)\,r = 0$; $D = \frac{v\lambda}{3}$; $S = \frac{v}{\lambda} \frac{\sigma_c}{\sigma_{sc}}$

4.) $J^{th} = D\, 4\pi R^2 \rho'(R)$

5.) for $\frac{dQ}{dr} \equiv 0$ $\rho(r) = \frac{Q}{S}\left(1 + c_1 \frac{e^{-r/A}}{r} + c_2 \frac{e^{+r/A}}{r}\right)$; $A = \frac{\lambda}{3}\sqrt{k^{th}}$

for $\sigma_c = 0.005$: $A = 43.5$ cm

6.) $J_0^{th} = 4\pi Q R A^2 (1 + R/A)$

7.) $J^{res} = 4\pi Q R B^2 (1 + R/B)$

8.) $\varepsilon_0 = \frac{A^2}{B^2} \frac{1 + R/A}{1 + R/B}$

9.) not black for thermal: $J = J_0 \varphi$ $\varphi < 1$

10.) for $R > 5$ cm $D(U) \frac{d^2(r\rho)}{dr^2} - S(U)\, r\rho = 0$ $D(U) = \frac{v\lambda(U)}{3}$

$S(U) = \frac{v}{\lambda(U)} \frac{\sigma_a(U)}{\sigma_{sc}(U)}$

$\rho(r) = \frac{c}{r}\left(e^{r/U} - e^{-r/U}\right)$

$U = \lambda(U) \sqrt{\frac{\sigma_{sc}(U)}{3\sigma_a(U)}}$

11.) $\varphi = \frac{\frac{\lambda(U)}{U}\left\{G - \frac{U}{R}\right\}}{\frac{\lambda}{R}(1 + R/A) + \frac{\lambda(U)}{U}\left\{G - \frac{U}{R}\right\}}$; $G = \frac{e^{R/U} + e^{-R/U}}{e^{R/U} - e^{-R/U}}$

for $R/U > 2$ $G \cong 1$ with less than 3.5% error.

12.) Lattice α absorbed in Graphite

$1-\alpha$ absorbed in Uranium

13.) $J^{th} = \alpha J^{th}$ approximation conservative:

$J^{res} = J^{res}$

$q = (1-\alpha) \frac{J^{th}}{J^{th} + J^{res}}$

or

$q = \frac{\varepsilon\alpha}{1 + \varepsilon\alpha}(1-\alpha)$ $\varepsilon = \varepsilon_0 \varphi$

14.) $\alpha_m = \frac{1 - q_m}{2}$ i.e. of those which are uselessly absorbed graphite absorbs ½ as thermal neutrons from which follows that the other half is absorbed by Uranium at resonance.

15.) $\varepsilon = \frac{4 q_m}{(1 - q_m)^2}$

16.) $\frac{4\pi R^3}{3} \Big/ V = \frac{1 - q_m}{6} \frac{R^2}{B^2} \frac{1}{1 + R/B}$

or $V = \frac{8\pi R B^2 (1 + R/B)}{1 - q_m}$

17.) Correction for lack of thermal neutron production near the spheres small we use

μq 0.9

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C - For spheres too small to apply the diffusion equation:

$$1-\varphi = \frac{1}{\pi R^2}\int_0^R e^{-a2\sqrt{R^2-r^2}}\,2\pi r\,dr \quad \cdots\cdots \quad (17)$$

a = absorption coefficient $= n\,\sigma_{abs}$

n = no of atoms/cm^3

$$1-\varphi = \frac{1}{2a^2R^2}\left[1-e^{-2aR}(1+2aR)\right] \quad \cdots\cdots \quad (18)$$

II) Cylindrical Case

D - Diffusion Equation:

$$D\left(\frac{\partial^2\rho}{\partial r^2}+\frac{1}{r}\frac{\partial\rho}{\partial r}\right) - S'\rho + Q = 0 \quad \cdots\cdots \quad (19)$$

$$I^{res} = 2\pi B^2 Z^* \frac{K_1(Z^*)}{K_0(Z^*)}\, Q \quad \cdots \quad (20) \qquad Z^* = \frac{R}{B}$$

$$\frac{V}{\pi R^2} = \frac{4}{1-\varphi_m}\,\frac{B^2}{R^2}\, Z^* \frac{K_1(Z^*)}{K_0(Z^*)} \quad \cdots\cdots \quad (21)$$

E - Cellular approx:

$$HR = \frac{\varphi}{1-\varphi}\left[\frac{Z\,K_1(Z)}{K_0(Z)}\right] \quad \cdots\cdots \quad (22)$$

$$Z_u = R/u$$

$$HR = \frac{\lambda_{tr}(U)}{\lambda_{tr}(C)}\, Z_u \frac{I_1(Z_u)}{I_0(Z_u)} \quad \cdots\cdots \quad (23)$$

$$q = \frac{2\pi A^2}{V}(Cyl)\left[1-\frac{2\pi B^2}{V} Z^* \frac{K_1(Z^*)}{K_0(Z^*)}\right] \quad \cdots\cdots \quad (24)$$

$$V = \pi(AY)^2 - \pi R^2$$

$$Cyl(H) = \frac{1}{\frac{1}{HR}+\frac{1}{Cyl(H\to\infty)}}$$

$$Cyl(H\to\infty) = Z\frac{I_1(z)}{I_0(z)}\;\frac{\frac{K_1(z)}{I_1(z)}-\frac{K_1(Y)}{I_1(Y)}}{\frac{K_0(z)}{I_0(z)}+\frac{K_1(Y)}{I_1(Y)}}$$

$$K_0(z) = -\left\{\gamma + \log \tfrac{1}{2}z\right\} I_0(z) + \sum_{p=1}^{\infty}\frac{(\frac{1}{2}z)^{2p}}{(p!)^2}\left\{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{p}\right\}$$

$$\gamma = 0.55$$

$$I_0(z) = 1+(\tfrac{1}{2}z)^2+\frac{(\frac{1}{2}z)^4}{(2!)^2}+\cdots\cdots$$

$$= \sum_{p=0}^{\infty}\frac{(\frac{1}{2}z)^{2p}}{(p!)^2}$$

F - Small Cylinders

$$\varphi(cyl) \approx \frac{3}{2}\varphi(sphere)$$

for the same radius — to a very rough approximation

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$V = \frac{\text{Total volume of } C}{\text{No of U spheres}}$

$$V = 4\pi \varphi \frac{\alpha}{q} A^2 R \left(1 + R/A\right) \quad \ldots\ldots (9)$$

For maximum q:

$$\alpha_m = \frac{1 - q_m}{2} \quad \ldots (10)$$

half of the wastefully absorbed neutrons

$$\varepsilon = \frac{4 q_m}{(1 - q_m)^2} \quad \ldots\ldots (11)$$

giving for the best C/U ratio

$$\frac{\frac{4\pi}{3} R^3}{V} = \frac{1 - q_m}{6} \frac{R^2}{B^2} \frac{1}{(1 + R/B)} \quad \ldots\ldots (12)$$

B - Cellular approximation

$$\frac{d\rho}{dr}(R) = H \rho(R) \quad \ldots\ldots (13)$$

$Z = R/A$

$$HR = (Z+1) \frac{\varphi}{1-\varphi} \quad \ldots\ldots (14)$$

$$\varphi = \frac{1}{(Z+1)\frac{1}{HR} + 1}$$

$$HR = \left(\frac{R}{U} \frac{e^{R/u} + e^{-R/u}}{e^{R/u} - e^{-R/u}} - 1\right) \frac{\lambda_{sc}(U)}{\lambda_{sc}(C)} \quad \ldots (15)$$

giving:

$$q = \frac{4\pi A^2 R}{V} [Cf] \frac{Q^{th}}{Q^{res}} \quad \ldots\ldots (16)$$

$$Cf = \text{Cell factor} = R \left(\frac{df}{dr}\right)_R$$

$$f(r) = 1 + C_1 e^{-r/A} + C_2 e^{r/A}$$

$$Cf = \frac{1}{\frac{1}{HR} + \frac{1}{Cf(H \to \infty)}}$$

$$Cf(H \to \infty) = 1 - \frac{Expr + 1}{Expr - 1} Z$$

$$Expr = \frac{1 + Y}{1 - Y} e^{-2(Y - Z)}$$

$$Y = \frac{\text{rad. of graphite sphere (cell)}}{A}$$

$$\frac{Q^{th}}{Q^{res}} = \left[1 - \frac{4\pi B^2 R (1 + R/B)}{V}\right]$$