The purpose of the present paper is to show that a chain reaction can be achieved by using an element like carbon for slowing down the neutrons in certain particular systems composed of uranium and carbon. The theory which is given in the present paper can be applied to elements other than carbon but it does not give any useful information for systems composed of uranium and hydrogen.

Though one might think that carbon should be much less efficient for slowing down neutrons than hydrogen from several points of view it would be preferable to use carbon in the form of graphite rather than hydrogen in the form of water. The capture cross-section of carbon for thermal neutrons $\sigma_c(C)$ is small. An upper limit of $\sigma_c(C) > 0.01 \times 10^{-24}$ cm^2 has been reported by Frisch, Halban and Koch, but this upper limit is not sufficiently low to allow us at present to conclude that a chain reaction could be maintained in homogeneous mixtures of uranium and carbon. For neutrons it takes about 6.5 collisions with carbon atoms to reduce their energy by a factor of e. Thus a neutron which is being slowed down by carbon stays for a long time within the resonance absorption region of uranium. Consequently, very low uranium concentrations would have to be used in order to avoid that a large fraction of the fast neutrons emitted by uranium is absorbed at resonance by uranium. At such very low uranium concentrations, on the other hand, the fraction of the thermal neutrons which is absorbed by carbon might perhaps be too large to permit a chain reaction.

It will be shown, however, in the present paper that if instead of using a homogeneous mixture of uranium and carbon a large number of spheres of uranium which may form, for instance, a close-packed hexagonal or cubic lattice are embedded in carbon, the ratio of the number of thermal neutrons and the number of resonance neutrons absorbed by the uranium can be so much increased that a chain reaction will become possible. It will be seen that this ratio strongly depends on the radius of the uranium spheres and that a rather small radius must be chosen in order to obtain most favorable conditions.

Single spheres of uranium .

We wish to first calculate ε , the ratio of the number of thermal neutrons, and the number of resonance neutrons which are absorbed by a single uranium sphere, which is embedded in an infinite space filled with carbon, in the special case in which the numbers of resonance neutrons and thermal neutrons produced per c.c. and second are equal and have the same value throughout the whole infinite mass of graphite. In order to obtain a conservative estimate for the value of ε we shall assume that all neutrons which have an energy between $E = 0.2\ E_0$ and $E = 2E$ where E_0 is the energy at which the resonance absorption of uranium has its maximum, are absorbed by uranium at resonance if they reach the surface of the uranium sphere by diffusion. That this is indeed a conservative assumption can be seen by considering an absorption line which obeys the Breit - Wigner formula and has its maximum at E_0 . For such an absorption line the absorption falls off with 1/v in the thermal ~~region~~ region. It reaches a minimum at 0.2 E_0 then it rises up to E_0 and falls again so rapidly that if E_0 is not too close to the thermal region the absorption becomes negligible for 2 E_0. If E_0 is higher than five volts and if the temperature of the thermal neutrons does not exceed 1/10 of a volt then the absorbing cross- section beyond 2 E_0 is less than 1/10 of the absorbing cross- section for the thermal neutrons.

A neutron which is slowed down by elastic collisions with carbon atoms and which enters the resonance at $E = E_2$ will survive on the average k^{res} collisions within the resonance region between E_2 and E_1, and we have

$$k^{res} = 6.5 \ln E_2/E_1 \tag{1}$$

Under the assumption which we have made above i.e. $E_2 = 2E_0$; $E_1 = 0.2 E_0$

$$k^{res} = 6.5 \ln 10 = 15$$

This may be compared with the average number of elastic collisions k^{th} which a thermal neutron will survive in carbon before being captured by a carbon atom. Since the capture cross- section of carbon is small compared with $\sigma_{sc}(C)$ the scattering cross - section of carbon for thermal neutrons the probability $g_1(h)$ that a thermal neutron will survive h collisions in carbon will be given by

$$g_1(h) = e^{-h\frac{\sigma_c(C)}{\sigma_{sc}(C)}}$$

Accordingly, k^{th} the average number of collisions which a thermal neutron will make with carbon atoms before being captured is given by

$$k^{th} = \int_0^\infty h\, g_1(h)\, dh$$

so that

$$k^{th} = \frac{\sigma_{sc}}{\sigma_c} \qquad (2)$$

Let us now first determine the number of thermal neutrons which are absorbed by a single uranium sphere of radius R embedded in an infinite space filled with carbon if Q thermal neutrons are produced per cc and sec. in the carbon. If R is large compared to $\lambda(C)$, themean free path for elasic scattering of thermal neutrons in carbon, the density ρ of the thermal neutrons in the carbon can be calculated by treating the problem as a diffusion phenomenon. We thus find for ρ as a function of the distance r from the center of the sphere

3 $$D(C)\frac{d^2(r\rho)}{dr^2} - S(C)\, r\rho + Q(r)\, r = 0$$

$$D(C) = \frac{v\lambda(C)}{3}, \quad S(C) = \frac{v\,\sigma_c(C)}{\lambda(C)\,\sigma_{sc}(C)}$$

If the same number of thermal neutrons are produced everywhere in the carbon per cc and sec. we have

4 $$\frac{dQ}{dr} \equiv 0$$

For a sphere which absorbes each thermal neutron which reaches its surface i.e. for a " black" sphere we have $\rho(R) = 0$ and from 3 and 4 find for $r > R$

5 $$\rho(r) = \frac{Q}{S(C)}\left(1 - \frac{R}{r}\, e^{-(r-R)/A}\right)$$

6 $$A = \sqrt{\frac{D}{S}} = \frac{\lambda(C)}{\sqrt{3}}\sqrt{\frac{\sigma_{sc}(C)}{\sigma_c(C)}} = \frac{\lambda(C)}{\sqrt{3}}\sqrt{k^{R}}$$

y^{th} the number of thermal neutrons which is absorbed by a single uranium sphere per sec. is given by

7 $$y^{th} = D(C)\, 4\pi R^2 \rho'(R)$$

and for a black uranium sphere we find from No. 5 $\quad y^{th} = y_0^{th}$

8 $$y_0^{th} = 4\pi Q R A^2 (1 + R/A)\ ; \quad A = \frac{\lambda(C)}{\sqrt{3}}\sqrt{k^{th}}$$

where A has the dimension of a length and will be called the range of thermal neutrons in carbon.

Quite similarly since we assume that the uranium sphere is "black" for resonance neutrons we can write for J^{res} the number of resonance neutrons absorbed by the sphere per second with good approximation

(9) $$J^{res} = 4\pi Q R B^2 (1 + R/B)$$

where

(10) $$B = \frac{\lambda^{res}}{\sqrt{3}} \sqrt{k^{res}}$$

is the mean free path of resonance neutrons for scattering and B has the dimension of a length and will be called the range of the resonance neutrons in carbon.

Expression No. 8 is identical with the expression No. 9 which holds for a sphere which is "black" for thermal neutrons. Only the values for λ the mean free path for scattering and k the average number of collisions which a neutron survives within the category called thermal or resonance are different for these two categories of neutrons. It would be strictly true that J the number of neutrons belonging to a category which will reach the sphere by diffusion per second is determined in the same way for different categories by λ the mean free path and $g(h)$ the function giving the probability of surviving h collisions with carbon atoms. In reality the function $g(h)$ is different for thermal and for resonance neutrons and expression No. 9 holds in so far as we may assume that J is determined with sufficient accuracy by λ and $k = \int_0^\infty h\, g(h)\, dh$ the first moment of $g(h)$.

From No. 8 and No. 9 we find as the value of ε for a sphere which is "black" for thermal neutrons $\varepsilon = \varepsilon_0$

(11) $$\varepsilon_0 = \frac{A^2}{B^2} \frac{1 + R/A}{1 + R/B}$$

Assuming, for example, $\sigma_{sc} = 4.8$; $\sigma_c(C) > 0.005$ and $\frac{\lambda^*(C)}{\lambda(C)} = 1.18$ we find for graphite of density 1.7:

$$A \geqq 43.5\,cm \quad B = 6.5\,cm$$

and so obtain for small values of R

$$\varepsilon_0(0) = \frac{A^2}{B^2} \geqq 45$$

and for large values of R

$$\varepsilon_0(\infty) = \frac{A}{B} \geqq 6.7$$

Large values of R correspond to plane layers of uranium and a comparison of these two values for ε_0 illustrates how very much superior small spheres of uranium are to plane layers.

A real sphere of uranium having a finite radius ~~below 8 centimeters~~ is not "black" for thermal neutrons and J^{th} the number of thermal neutrons absorbed per second by the sphere is smaller than J_0^{th}. We write

(12) $$J^{u} = J_0^{u}\varphi$$

and accordingly we have

(13) $\varepsilon = \varepsilon_0 \varphi$ and (14) $\varepsilon = \frac{A^2}{B^2}\,\frac{1+R/A}{1+R/B}$

In order to calculate φ we take into account that inside the uranium sphere the thermal neutron density ρ obeys the equation

(15) $$D(u)\frac{d^2(r\rho)}{dr^2} - \delta(u)\,r\rho = 0$$

$$D(u) = \frac{v\lambda(u)}{3} \qquad \delta(u) = v N_u \sigma_a(u)$$

having as its solution

(16) $$\rho(r) = \frac{C}{r}\left(e^{r/u} - e^{-r/u}\right); \quad r<R$$

(17) where $u = \sqrt{\frac{D}{\delta}}$ ~~. For pure uranium we have~~ $= \sqrt{\frac{\lambda(u)}{3 N_u \sigma_a(u)}}$

and for [redacted] pure uranium metal we have

(18) $$u = \lambda(u)\sqrt{\frac{\sigma_{sc}(u)}{3\sigma_a(u)}}$$

From equations 3, 4 and 16 we find that J^{u} the number of thermal neutrons diffusing into the sphere per second is given by

$$J^{u} = J_0\varphi$$

where

(19) $$\varphi = \frac{\frac{\lambda_{sc}(u)}{u}\left\{\frac{e^{R/u}+e^{-R/u}}{e^{R/u}-e^{-R/u}} - \frac{u}{R}\right\}}{\frac{\lambda_{sc}(C)}{R}\left(1+\frac{R}{A}\right) + \frac{\lambda_{sc}(u)}{u}\left\{\frac{e^{R/u}+e^{-R/u}}{e^{R/u}-e^{-R/u}} - \frac{u}{R}\right\}}$$

For uranium in its pure state we have from No. 14, 18, and 19

$$(20a)\qquad \varepsilon = \left\{\frac{A^2}{B}\right\} \times \left\{\frac{1}{1+R/B} \times \frac{1}{\left(\dfrac{\lambda(C)}{RG\sqrt{\dfrac{3\sigma_a(U)}{\sigma_s(U)}} - \lambda(U)} + \dfrac{1}{1+R/A}\right)}\right\}$$

where G stands for

$$G = \frac{e^{R/U} + e^{-R/U}}{e^{R/U} - e^{-R/U}} \quad ; \quad R/U = \frac{R}{\lambda(U)}\sqrt{\frac{3\sigma_a(U)}{3\sigma_s(U)}}$$

For $R/U > 2$ we can write $G \cong 1$, the difference being about 3.5% for $R/U = 2$.

The first factor in expression No. 20 a increases proportionately with the reciprocal value of the capture cross-section of carbon. The second factor is practically independent of the carbon cross-section, since we have $R/A << 1$. Its value is determined by the density of graphite and uranium and the nuclear values $\sigma_s(U)$, $\dfrac{\sigma_a(U)}{\sigma_s(U)}$. The value of R may be so chosen as to make this factor a maximum.

All expressions for ε were so far obtained from diffusion equations involving the assumptions

$$R >> \lambda(C)\ ; \quad R >> \lambda(U)\ ; \quad U >> \lambda(U)$$

For small values of R the problem can no longer be treated as a diffusion phenomenon and we shall, therefore, refrain from using expression No. 20 or 20 a for values of R of less than 5 cm.

Single spheres of uranium .

We wish to first calculate ξ , the ratio of the number of thermal neutrons, and the number of resonance neutrons which are absorbed by a single uranium sphere which is embedded in an infinite space filled with carbon, in the special case in which the numbers of resonance neutrons and thermal neutrons produced per c.c. and second are equal and have the same value throughout the whole infinite mass of graphite. ~~In~~ order to obtain ~~a conservative~~ an estimate for the value of ξ we shall assume that all neutrons which have an energy between (same energy) $E_1 = 0.2\ E_0$ and $E_2 = 2E_0$ (another 10 E_0) ~~where E_0 is the energy at which the resonance absorption of uranium has its maximum,~~ are absorbed by uranium at resonance if they reach the surface of the uranium sphere by diffusion. That this is indeed a conservative assumption ~~can be seen by considering an~~ for a single absorption line which obeys the Breit - Wigner formula ~~and has its maximum at E_0~~ . For such an absorption line the absorption falls off with $1/v$ in the thermal region. It reaches a minimum at $0.2\ E_0$ then it rises up to E_0 and falls again so rapidly that if E_0 is not too close to the thermal region the absorption becomes negligible for $2\ E_0$. (If E_0 is higher than five volts and if the temperature of the thermal neutrons does not exceed 1/10 of a volt then the absorbing cross- section beyond $2\ E_0$ is less than 1/10 of the absorbing cross- section for the thermal neutrons.)

A neutron which is slowed down by elastic collisions with carbon atoms and which enters the resonance at $E = E_2$ will survive on the average k^{res} collisions within the resonance region between E_2 and E_1 and we have

$$k^{res} = 6.5 \ln E_2/E_1 \qquad (1)$$

$E_2 = 10\,E_1$

Under the assumption which we have made above i.e. ~~$E_2 = 2E_0$; $E_1 = 0.2\,E_0$~~

$$k^{res} = 6.5 \ln 10 = 15$$

This may be compared with the average number of elastic collisions k^{th} which a thermal neutron will survive in carbon before being captured by a carbon atom. Since the capture cross- section of carbon is small compared with $\sigma_{sc}(C)$ the scattering cross - section of carbon for thermal neutrons the probability $g_1(h)$ that a thermal neutron will survive h collisions in carbon will be given by

$$g_1(h) = e^{-h\frac{\sigma_c(C)}{\sigma_{sc}(C)}}$$

Accordingly, k^{th} the average number of collisions which a thermal neutron will make with carbon atoms before being captured is given by

$$k^{th} = \int_0^\infty h\, g_1(h)\, dh$$

so that

(2) $$k^{th} = \frac{\sigma_{sc}}{\sigma_c}$$

Quite similarly since we assume that the uranium sphere is "black" for resonance neutrons we can write for J^{res} the number of resonance neutrons absorbed by the sphere per second with good approximation

(9) $$J^{res} = 4\pi Q R B^2 (1 + R/B)$$

where

(10) $$B = \frac{\lambda^{res}}{\sqrt{3}} \sqrt{k^{res}}$$

is the mean free path of resonance neutrons for scattering and B has the dimension of a length and will be called the range of the resonance neutrons in carbon.

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blocks

A real sphere of uranium having a finite radius ~~below 8 centimeters~~ is not "black" for thermal neutrons and the number of thermal neutrons absorbed by the sphere per second is smaller than J_0^{th}. We write

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and accordingly we have

(13) $$\varepsilon = \varepsilon_0\varphi \quad \text{and (14.)} \quad \varepsilon = \frac{A^2}{B^2}\,\frac{1+R/A}{1+R/B}\,\varphi$$

In order to calculate φ we take into account that inside the uranium sphere the thermal neutron density ρ obeys the equation

(15) $$D(U)\frac{d^2(r\rho)}{dr^2} - \delta(U)\,r\rho = 0$$

$$D(U) = \frac{v\lambda(U)}{3}\ ; \quad \delta(U) = v N_u\,\sigma_a(U)$$

having as its solution

(16) $$\rho(r) = \frac{C}{r}\left(e^{r/U} - e^{-r/U}\right);\quad r<R$$

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and for ~~[illegible]~~ pure uranium metal we have

(18) $$U = \lambda(U)\sqrt{\frac{\sigma_{sc}(U)}{3\sigma_a(U)}}$$

From equations 3, 4 and 16 we find that J^{U} the number of thermal neutrons diffusing into the sphere per second is given by

$$J^{U} = J_0\varphi$$

where

(19) $$\varphi = \frac{\frac{\lambda_{sc}(U)}{U}\left\{\frac{e^{R/U}+e^{-R/U}}{e^{R/U}-e^{-R/U}} - \frac{U}{R}\right\}}{\frac{\lambda_{sc}(C)}{R}\left(1+\frac{R}{A}\right) + \frac{\lambda_{sc}(U)}{U}\left\{\frac{e^{R/U}+e^{-R/U}}{e^{R/U}-e^{-R/U}} - \frac{U}{R}\right\}}$$

Single spheres of uranium.

We wish to first calculate ε the ratio of the number of thermal neutrons, and the number of resonance neutrons which are absorbed by a single uranium sphere, which is embedded in an infinite space, filled with carbon, in the special case in which the number of resonance neutrons and thermal neutrons produced per c.c. and second are equal and have the same value throughout the whole infinite mass of graphite. In order to obtain a conservative estimate for the value of ε we shall assume that all neutrons which have an energy between $E = 0.2\ E_0$ and $E = 2E_0$ where E_0 is the energy at which the resonance absorption of uranium has its maximum, are absorbed by uranium at resonance if they reach the surface of the uranium sphere by diffusion. That this is indeed a conservative assumption can be seen by considering an absorption line which obeys the Breit-Wigner formula and has its maximum at E_0. For such an absorption line the absorption falls off with 1/v in the thermal region. It reaches a minimum at $0.2\ E_0$ then it rises up to E_0 and falls again so rapidly that if E_0 is not too close to the thermal region the absorption becomes negligible for $2\ E_0$. If E_0 is higher than five volts and if the temperature of the thermal neutrons does not exceed 1/10 of a volt then the absorbing cross-section beyond $2\ E_0$ is less than 1/10 of the absorbing cross-section for the thermal neutrons.

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(1)

Under the assumption which we have made above i.e. $E_2 = 2E_0$; $E_1 = 0.2\ E_0$

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This may be compared with the average number of elastic collisions k which a thermal neutron will survive in carbon before being captured by a carbon atom. Since the capture cross-section of carbon is small compared with $\sigma_{sc}(C)$ the scattering cross-section of carbon for thermal neutrons the probability $g_1(h)$ that a thermal neutron will survive h collisions in carbon will be given by

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